

# Robust Control Design Based on Differential Evolution for Two-Mass System

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**Abstract**— This paper presents a robust control design based on constrained optimization using Differential Evolution (DE). The feedback controller is designed based on state space model of the plant considering structured uncertainty such that the closed-loop system would have maximum stability radius. A wedge region is assigned as a constraint for desired closed loop poles location. The proposed control technique is applied to a two-mass system that is known as benchmark problem for robust control design. The simulation results seem to be interesting in which the robustness performance is achieved in the presence of parameter variations of the plant.

**Keywords**— robust control, two-mass system, differential evolution

## I. INTRODUCTION

Robustness has been an important issue in control systems design. A successfully designed control system should be always able to maintain stability and performance level in spite of uncertainties in system dynamics including parameter variations of the plant.

In robust control theory,  $H_\infty$  optimization approach and the  $\mu$ -synthesis/analysis method are well developed and elegant [1]. They provide systematic design procedures of robust controllers for linear systems. However, the mathematics behind the theory is quite involved. It is not straightforward to formulate a practical design problem into  $H_\infty$  or  $\mu$  design framework.

In this paper, we propose an alternative technique of robust feedback control design via constrained optimization. We employ DE (differential evolution) as a modern evolutionary algorithm that is fast and reasonably robust for optimization.

To deal with the plant's parameter uncertainty, we employ complex stability radius as a tool of measuring system robustness. In addition, the desired response is automatically defined by assigning a regional closed loop poles placement. This region will be incorporated in the DE-based optimization as a constraint. In other word, the controller design technique is based on a constrained optimization to

obtain a set of feedback controller gains such that the closed-loop system would have maximum complex stability radius.

At the end of this paper, we will present the simulation results of our proposed control design for two-mass system which is commonly known as a benchmark problem for robust control design [2-6].

## II. BRIEF REVIEW

### A. Problem Statement

Consider a plant model of linear time-invariant continuous-time system in state space form:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}\quad (1)$$

where  $x \in R^n$ ,  $u \in R^m$  and  $y \in R^p$  are state vector, control input and output vector respectively. It is assumed that the system given in (1) is completely state controllable and all state variables are available for feedback. One can use state feedback controller with feed-forward integral gain ( $k_i$ ) as shown in Fig. 1. The controller gains ( $k := (k_1, k_2, k_3, \dots, k_n)$  and  $k_i$ ) can be computed based on classical methods such as pole placement or linear quadratic optimal control via Riccati equation. These methods assume of course that there is no plant uncertainty.

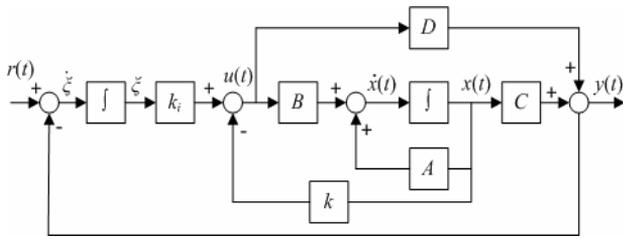


Fig. 1. State feedback controller with feed-forward integral gain

In this work, constrained optimization using DE is employed to find a set of robust controller gains so that the plant uncertainty is automatically handled with the use of stability radius that will be discussed in the next section. In addition, a region of closed loop poles is incorporated as optimization constraint to allow the designers to define the desired control performance.

### B. Stability radius

In this section, a tool of measuring system robustness is presented. It is called as stability radius [7]. It is a maximum distance to instability. Equivalently, a system having a larger stability radius implies that the system can tolerate more perturbations. In general, we can classify stability radius into two types; complex stability radius and real stability radius. Compared to real stability radius, complex stability radius can handle a wider class of perturbations including nonlinear, linear-time-varying, nonlinear-time-varying and nonlinear-time-varying-and-dynamics perturbations [8]. For this reason we use complex stability radius in this work. The complex stability radius will be maximized as in the optimization.

The definition of complex stability radius is given here. Let  $C$  denote the set of complex numbers.  $C_- = \{z \in C | \text{Real}(z) < 0\}$  and  $C_+ = C \setminus C_-$  is the closed right half plane. Consider a nominal system in the form:

$$\dot{x}(t) = Ax(t) \quad (2)$$

$A(t)$  is assumed to be stable. The perturbed open-loop system is assumed as:

$$\dot{x}(t) = (A(t) + E\Delta(t)H)x(t) \quad (3)$$

where  $\Delta(\cdot)$  is a bounded time-varying linear perturbation.  $E$  and  $H$  are scale matrices that define the structure of the perturbations. The perturbation matrix itself is unknown. The stability radius of (3) is defined as the smallest norm of  $\Delta$  for which there exists a  $\Delta$  that destabilizes (2) for the given perturbation structure  $(E, H)$ .

For the controlled perturbed system in the form (2), let:

$$G(s) = H(sI - A)^{-1}E \quad (4)$$

be the “transfer matrix” associated with  $(A, E, H)$ , then the complex stability radius is defined by the following definition.

**Definition 1:** [7] The complex stability radius,  $r_c$ :

$$r_c(A, E, H, C_+) = \left[ \max_{s \in \partial C_+} \|G(s)\| \right]^{-1} \quad (5)$$

where  $\partial C_+ = \overline{C_-} \cap C_+$  is the boundary of  $C_+$ . In other words, a maximum  $r_c$  can be achieved by minimizing the  $H_\infty$  norm of the “transfer matrix”  $G$  [8].

## III. DE-BASED CONTROL DESIGN

### A. Brief Overview of DE

A DE algorithm is a stochastic search optimization method that is fast and reasonably robust. DE is capable of handling non-differentiable, non-linear, and multimodal objective functions [9]. DE is one of the most promising novel evolutionary algorithms for solving global optimization problems [10]. It was proposed by Storn and Price not long ago in 1995 [11].

The structure of DE is similar to other evolutionary algorithms. The first generation is initialized randomly and further generations evolve by applying the evolutionary operators: mutation, recombination and selection to every population member until a stopping criterion is satisfied. There are some variants of DE. The DE variant called DE/rand/1/bin [12] is used here.

Furthermore, there are only few parameters defined by user in DE. Similar to other evolutionary algorithms, users have to select number of population,  $NP$ . The other control parameters are  $F$  (mutation scaling factor) and  $CR$  (crossover rate factor) which are normally valued between  $[0, 1]$ . Users can refer to [13] for the choice of these parameters.

### B. Constrained Optimization

The objective of the optimization is to maximize the complex stability radius ( $r_c$ ), however we will convert into minimization mode in this work by putting negative sign. Based on our approach, the searching procedure of the robust controller gains using constrained optimization can be formulated as follows (Table 1).

Table 1. Constrained optimization

Minimize:	$f(X) = -r_c(X)$
Subject to constraint:	$\lambda_n(X) \in \psi \quad \text{for } n=1,2,\dots$
and boundary constraint:	$X \in [l_b, u_b]$

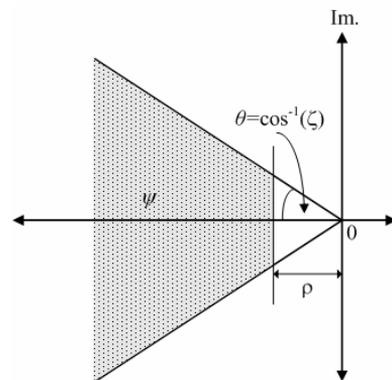


Fig. 2. A wedge region in complex plane for closed loop poles placement

$X = K = (k_1, k_2, \dots, k_n, k_i)$  is the vector solutions such that  $X \in S \subseteq R^{n+1}$ .  $S$  is the search space, and  $F \subseteq S$  is the

feasible region or the region of  $S$  for which the constraint is satisfied. The constraint here is the closed loop poles region; in the feasible region, the controller gains are found such that the closed loop poles ( $\lambda$ ) lie within a wedge region ( $\psi$ ) of a complex plane as given in Fig. 2. The wedge region can be specified by two parameters  $\theta$  and  $\rho$  which are related to desired transient response characteristics i.e.: damping ratio ( $\zeta$ ) and settling time.

### C. Constraint Handling

An efficient and adequate constraint-handling technique is a key element in the design of stochastic algorithms to solve complex optimization problems. Although the use of penalty functions is the most common technique for constraint-handling, there are a lot of different approaches for dealing with constraints [14].

Instead of using penalty approach like in [15] where the optimizer seemed to be inefficient (high iterations), we adopt a dynamic-objective constraint-handling method (DOCHM) [16] in order to improve the efficiency. Through defining distance function  $F(X)$ , DOCHM converts the original problem into bi-objective optimization problem  $\min(F(X), f(X))$ , where  $F(X)$  is treated as the first objective function and  $f(x)$  is the second (main) one.

The auxiliary distance function  $F(X)$  will be merely used to determine whether or not an individual (candidate of solution) is within the feasible region and how close a particle is to the feasible region. If an individual lies outside the feasible region (at least an eigenvalue lies outside the wedge region), the individual will take  $F(X)$  as its optimization objective. Otherwise, the individual will instead optimize the real objective function  $f(X)$ . During the optimization process if an individual leaves the feasible region, it will once again optimize  $F(X)$ . Therefore, the optimizer has the ability to dynamically drive the individuals into the feasible region.

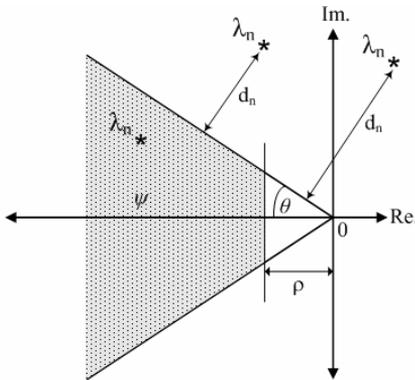


Fig. 3. Eigenvalue distance to the wedge region in complex plane

The procedure of the DOCHM applied to the eigenvalue assignment in the wedge region is illustrated in the following pseudo-code (Table 2). Referring to Fig. 3, let  $d_n$  is an outer distance of an eigenvalue ( $\lambda_n$ ) to the wedge region. It is noted that if an eigenvalue lies within the wedge region,  $d_n=0$ .  $F(X)$  is defined by:

$$F(X) = \sum_{i=1}^{n+1} \max(0, d_n(\lambda_n(X))) \quad (6)$$

Table 2. Pseudo-code for constraint handling

If	$F(X) = 0$
	$f(X) = -r_c(X)$
Else	$f(X) = F(X)$
End	

### D. Stopping criterion

In literatures, mostly two stopping criteria are applied in single-objective optimization: either an error measure if the optimum value is known is used or the number of function evaluations (number of iterations). There are some drawbacks for both. The optimum has to be known in the first method, so it is generally not applicable to real-world problems because the optimum is usually not known a priori. The second method is highly dependent on the objective function. Because generally no correlation can be seen between an optimization problem and the required number of function evaluations (iterations), it has to be determined by trial-and-error methods usually. Improper selection of the number of iterations to terminate the optimization can lead to either premature convergence or expensive optimization runs (excessive computational effort).

As a result, it would be better to use stopping criterion that consider knowledge from the state of the optimization run. The time of termination would be determined adaptively, so the optimization run would be efficient. Several stopping criterions are reviewed in [17]. Although the authors did not conclude which one is the best for all problems, it is believed that performance improvement can be obtained with adaptive stopping criterion.

In this work, we adopt the stopping criterion which is distribution-based criterion which considers the diversity in the population. If the diversity is low, the individuals are close to each other, so it is assumed that convergence has been obtained [17]. Standard deviation ( $\sigma$ ) of the best individuals in each dimension during iterations is checked. If it is below a threshold  $\mathcal{E}$  (small number) for sufficiently large number of iterations  $\eta$ , the optimization will be terminated. It can be formulated as in Table 3; where  $x_{best,d}^j$  represents the best individual in  $j$ -th generation (iteration) for  $d$  dimension.

Table 3. Stopping criterion

If	$\sigma_d = \sqrt{\frac{1}{\eta} \sum_{j=1}^{\eta} (x_{best,d}^j - \bar{x}_{best,d})^2} < \mathcal{E}(\max(x_{best,d}) - \min(x_{best,d}))$
	(for $d=1, 2, \dots, D$ )
	stop iterations.
End	

## IV. ROBUST CONTROL DESIGN FOR TWO-MASS SYSTEM

In this section, an illustrative example of the proposed method to two-mass system is presented. This system has been used as benchmark problem for robust control design

[5]. Consider the two-mass system shown in the Fig. 4. A control force ( $u$ ) acts on body 1 and the position of body 2 is measured. Both masses are equal to one unit ( $m_1=m_2=1$ ) and the spring constant is assumed to be in the range  $0.5 \leq k \leq 2$ . The system can be represented in state space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & 0 & 0 \\ k & k & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \quad (12)$$

where:  $x_1$ : position of mass-1  
 $x_2$ : position of mass-2  
 $x_3$ : velocity of mass-1  
 $x_4$ : velocity of mass-2

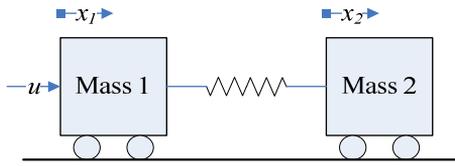


Fig. 4. Two-mass and spring system

The plant uncertainty is due to variations of the spring constant where the nominal value is selected for the worst case of  $k=0.5$ . Therefore uncertainties appear in the rows 3-4 and the columns 1-2 of the state matrix. The scale matrices as the perturbation structure for the closed loop system are  $E_{cl}$  and  $H_{cl}$  whose diagonal elements in rows 3-4 of  $E_{cl}$  and in columns 2-3 of  $H_{cl}$  are respectively equal to 1.

$$E_{cl} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad H_{cl} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The next is to choose the parameters of the wedge region (Fig. 2) whose role is to locate the closed loop poles. The damping ratio is usually set to  $\zeta=0.7$  to produce sufficient overshoot damping in the response. The transient margin ( $\rho$ ) is specified according to the desired speed of the response. This is problem-dependent parameter. Here, we set  $\rho=1$ . In addition, the main DE-based optimization parameters are listed in Table 4.

Table 4. DE-based optimization parameters

Parameter	Symbol	Value
Dimension of the problem	$D$	5
Population size	$NP$	100
Mutation scaling constant	$F$	0.9
Crossover rate constant	$CR$	0.5
Upper and lower bounds of solution	$\pm BD$	$\pm 50$
Maximum iteration	$j_{max}$	2000
Number of iteration for which stopping criterion applies	$\eta$	200
Standard deviation threshold for which stopping criterion applies	$\varepsilon$	1%

## V. RESULTS

The optimization run has been performed in MATLAB 2006. Since DE is a stochastic optimization, a number of optimization runs need to be executed with different initial random seeds. To get an optimal solution and to evaluate the quality of the solution (robustness, convergence, repeatability), 15 runs have been executed here. The mean value, the standard deviation of the fitness value ( $f(X)=-r_c$ ) and other results are recorded in Table 5.

Table 5. Optimization results for 15 runs

Average $f(X)$	-0.32
Median $f(X)$	-3.07
Standard deviation $f(X)$	0.005
Range of $f(X)$	-0.31 to -0.33
Average number of iteration	715
Average computation time	0.77 minutes

Table 6. Controller gains for two-mass system

Controller gains	$k_1$	$k_2$	$k_3$	$k_4$	$k_5$
	18.49	19.04	47.27	7.30	-10.54

Form Table 5, it can be seen that the optimization results in a robust solution with a small standard deviation, the range of the fitness value is also very small. This means that the optimizer has a good repeatability property. The distribution of eigenvalues for those 15 runs can be seen in Fig.5. All eigenvalues lie within the specified wedge region.

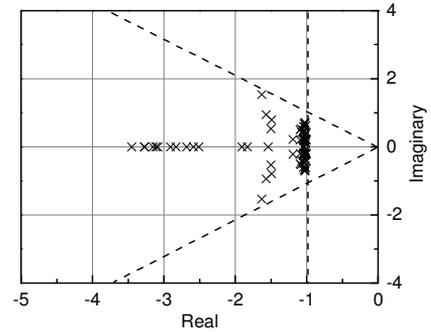


Fig. 5. . Distribution of eigenvalues within the wedge region for 15 runs

Furthermore, to see the controller performance, a set of controller gains is picked from the median data and it is shown in Table 6. Fig. 6 shows 10 random samples (with 10 random values of the spring constant  $0.5 \leq k \leq 2$ ) of step response (position of mass-2) with the proposed DE-based feedback controller (DEFC). For comparison, two conventional LQR-based controllers (linear quadratic regulator) are also designed with the following sets of  $Q$  and  $R$  matrices respectively for LQR1 and LQR2:

$$Q_1 = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 100 \end{bmatrix} \quad \text{and} \quad R_1 = [1]$$

$$Q_2 = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and } R_2 = [10].$$

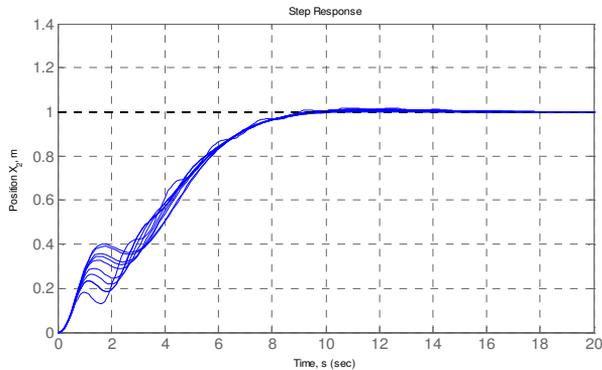


Fig. 6. Ten random samples of step response for DEFC

Figs. 7-8 show 10 random samples of the step response (the position of mass-2) for LQR1 and LQR2 respectively. It can be seen that the proposed controller (DEFC) is more robust compared to the two LQR-based controller (LQR1 and LQR2) designed in this work.

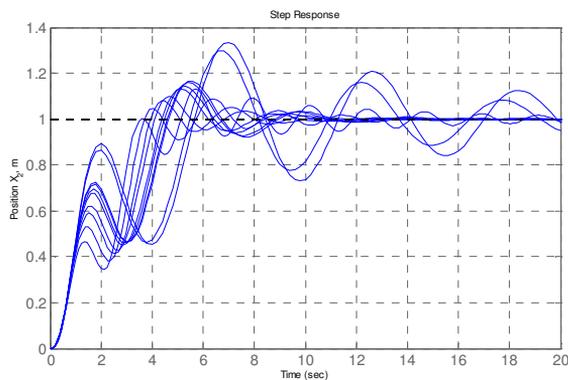


Fig. 7. Ten random samples of step response for LQR1

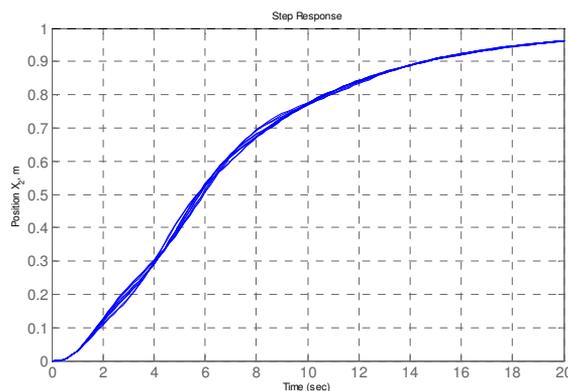


Fig. 7. Ten random samples of step response for LQR2

## VI. CONCLUSIONS AND DISCUSSIONS

A robust state feedback control design via constrained optimization using DE has been presented. The designed controller has shown a robust performance in the presence of parameter variations. The DE-based constrained optimization effectively locates closed loop poles within a prescribed wedge region and able to maximize the stability radius with a good repeatability of the solution.

Finally, this paper reports a preliminary research. To further evaluate the robust performance of the proposed controller, it is necessary to analyze in more detail using robust control concepts. Comparison with other robust control techniques is also necessary. This will be done in the future.

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