New Properties for Certain Generalized Ces’aro Integral Operator

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Abstract— In this work, we obtain the order of convexity of the integral operator which is a generalization to Ces’aro integral operator. Furthermore, some other properties of the integral operator by using the concept of the norm and pre-Schwarzian derivatives are obtained.

Keywords— Analytic function; pre-Schwarzian derivatives; Ces’aro integral operator; starlike function; convex function.

I. INTRODUCTION

The Ces’aro operator $C$ acts formally on the power series $f(z) = \sum_{k=0}^{\infty} a_k(z^k)$ as

$$C[f](z) = \int_0^1 \frac{f(t)}{1-t} \, dt$$

the classical Ces’aro means play an important role in geometric function theory (see [2], [3],[4],[5]). Let $H$ denote the class of all analytic functions in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$ of complex plane.

Let $A$ denote the class of functions $f \in H$ normalized by $f(0) = 0$, $f'(0) = 1$.

Also, let $S$ denote the class of all univalent functions in $A$.

A function $f$ belonging to $A$ is said to be starlike of order $\alpha$ in $U$ if it satisfies

$$f \in S(\alpha) \iff \Re \{ \frac{zf'(z)}{f(z)} \} > \alpha, \quad (z \in U),$$

for some $0 \leq \alpha < 1$.

Further, a function $f$ belonging to $A$ is said to be convex in $U$ if it satisfies

$$f \in K(\alpha) \iff \Re \{ \frac{zf''(z)}{f'(z)^2} + 1 \} > \alpha, \quad (z \in U),$$

for some $0 \leq \alpha < 1$.

A function $f$ belonging to $A$ is said to be the class $R(\alpha)$ iff

$$\Re \{ f'(z) \} > \alpha, \quad (z \in U),$$

for some $0 \leq \alpha < 1$.

Very recently, Frasin and Jahangiri [6] defined the family $B(\mu, \alpha)$, for some $(0 \leq \mu, 0 \leq \alpha < 1)$, so that it consists of functions $f \in A$ satisfying the condition

$$\left| f'(z) \left( \frac{z}{f(z)} \right)^\mu - 1 \right| < 1 - \alpha, \quad (z \in U)$$

The family $B(\mu, \alpha)$ is a comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well known ones. For example,

$$B(1, \alpha) = S'(\alpha), \quad \text{and} \quad B(0, \alpha) = R(\alpha).$$

Another interesting subclass is the special case $B(2, \alpha) = B(\alpha)$, which has been introduced by Frasin and Darus [7].

Let $f : U \to \mathbb{C}$ be analytic and locally univalent. The pre-Schwarzian derivative (or nonlinearity) $T_f$ to $f$ is defined by

$$T_f = \frac{f''}{f'}.$$ 

Also, with respect to the Hornich operation, the quantity

$$\left\| T_f \right\| = \sup_{z \in U} \left| T_f(z) \right|.$$

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can be regarded as a norm on the space of uniformly locally univalent analytic functions $f \in U$.

It is known that $T_f < \infty$ if and only if $f$ is uniformly locally univalent.

It is well-known that from Becker’s univalence criterion [8]: every analytic function $f$ in $U$ with $\|T_f\| \leq 1$ is in fact univalent in $U$. Conversely, $\|T_f\| \leq 6$ holds if $f$ univalent.

Consider the general integral operator defined by the formula:

$$C(f_1, f_2, \ldots, f_m)_{\beta_1, \beta_2, \ldots, \beta_m}(z) =$$

$$\frac{1}{\beta_1} \int_0^z \frac{f_1(t)}{1-t} \frac{f_2(t)}{1-t} \cdots \frac{f_m(t)}{1-t} \, dt, \quad (z \in U \setminus \{0\}), \quad (1.3) ,$$

where $\beta_i \in \mathbb{C} \setminus \{0\}, \forall i = 1, \ldots, m$, and the functions $f_i(z)$ are in $B(\mu, \alpha)$. It is clear that when $\beta_i = 1$ and $\beta_j = 0, j = 2, \ldots, m$ the integral operator (1.3) reduces to Cesàro integral operator (1.1).

In this paper we will study some general properties for function $zC(f_1, f_2, \ldots, f_m)_{\beta_1, \beta_2, \ldots, \beta_m}(z) =

\frac{1}{\beta_1} \int_0^z \frac{f_1(t)}{1-t} \frac{f_2(t)}{1-t} \cdots \frac{f_m(t)}{1-t} \, dt, (z \in U \setminus \{0\}) , \quad (1.3) ,$

For the purpose this work, we shall make use of the following lemmas.

Lemma 1.1 [1]

Let the analytic function $f$ be regular in the disk with $f(0) = 0$. If $|f(z)| \leq 1$, for all $(z \in U)$ then $|f(z)| \leq |z|$, $(z \in U)$. The equality can hold only if $f(z) = ez$, where $|e| = 1$.

Lemma 1.2 Let the analytic and locally univalent $f$ in $U$. Then

(i) If $\|T_f\| \leq 1$, then $f$ is univalent , and

(ii) If $\|T_f\| \leq 2$, then $f$ is bounded .

The part (i) is due to Becker [8] and sharpness of the constant $1$ is due to Becker and Pommerening [9]. The part (ii) is obvious (see [10], Corollary 2.4). Note also that, recently, Kari and Per Hag [12] gave a necessary and sufficient condition for $f \in S$ to have a John disk as the image in terms of the preSchwarzian derivative of $f$.

Also, the norm estimates for typical subclasses of univalent functions are investigated by many authors. See for example ([10], and so on).

Lemma 1.3 [11]

Let $0 \leq \alpha < 1$ and $f \in S$.

(i) If $f$ is starlike of order $\alpha$, then $\|T_f\| \leq 6 - 4\alpha$, and

(ii) If $f$ is convex of order $\alpha$, then $\|T_f\| \leq 4(1 - \alpha)$.

The constants are sharp.

II. MAIN RESULTS

Theorem 2.1

Let $f_i \in \mathcal{A}$ be in the class $B(\mu, \alpha)$, $\mu \geq 0$, $0 \leq \alpha < 1$, for all $i = 1, 2, \ldots, m$. If $|f_i(z)| \leq M$, $0 \leq |z| < \frac{1}{2}$, $(M \geq 1, z \in U)$, then

$$zC(f_1, f_2, \ldots, f_m)_{\beta_1, \beta_2, \ldots, \beta_m}(z) =$$

$$\int_0^z \frac{f_1(t)}{1-t} \frac{f_2(t)}{1-t} \cdots \frac{f_m(t)}{1-t} \, dt ,$$

is convex of order $\delta$,

where

$$\delta = 1 - \sum_{i=1}^m \frac{1}{\beta_i}((2 - \alpha)M^{\beta_i-1} + 1),$$

and

$$\sum_{i=1}^m \frac{1}{\beta_i}((2 - \alpha)M^{\beta_i-1} + 1) < 1, \quad \beta_i \in \mathbb{C} \setminus \{0\}, \quad \text{for all } i = 1, 2, \ldots, m .$$

Proof:

From the definition of the operator (1.3) we have

$$zC(f_1, f_2, \ldots, f_m)_{\beta_1, \beta_2, \ldots, \beta_m}(z) = \prod_{i=1}^m \frac{f_i(t)}{1-t} \, dt ,$$

For $f_i \in B(\mu, \alpha)$. It is easy to see that

$$(zC(f_1, f_2, \ldots, f_m)_{\beta_1, \beta_2, \ldots, \beta_m}(z))' =$$

$$\prod_{i=1}^m \frac{f_i'(t)}{1-t} \cdot \text{ (2.1).}$$

Differentiating both sides of (2.1) logarithmically, we obtain

$$\frac{z(zC(f_1, f_2, \ldots, f_m)_{\beta_1, \beta_2, \ldots, \beta_m}(z))^n}{(zC(f_1, f_2, \ldots, f_m)_{\beta_1, \beta_2, \ldots, \beta_m}(z))'} =$$

$$\prod_{i=1}^m \frac{f_i'(t)}{1-t} \cdot \text{ (2.2).}$$

\begin{align}
\sum_{i=1}^m \frac{1}{\beta_i} \left( \frac{f_i'(z)}{f_i(z)} + \frac{1}{1-z} \right) ,
\end{align}

which readily shows that

\begin{align}
\frac{z(zC(f_1, f_2, \ldots, f_m)_{\beta_1, \beta_2, \ldots, \beta_m}(z))^n}{(zC(f_1, f_2, \ldots, f_m)_{\beta_1, \beta_2, \ldots, \beta_m}(z))'} \leq \sum_{i=1}^m \frac{1}{\beta_i} \left[ |z f_i'(z)| + \frac{z}{|1-z|} \right] ,
\end{align}

\begin{align}
\sum_{i=1}^m \frac{1}{\beta_i} \left( |f_i'(z)| \left( \frac{z}{z-1} \right)^{n-1} \right) + \frac{z}{|1-z|} .
\end{align}
Since \(|f_i(z)| \leq M, (z \in U, i \in \{1, 2, \ldots, m\})\), applying the Schwarz lemma, we obtain
\[|f_i(z)| \leq M, (z \in U, i \in \{1, 2, \ldots, m\}).\]
Therefore, from (2.2), we obtain
\[z(zf_i(z), \ldots, f_m(z)) \leq \left|z \cdot 
\sum_{i=1}^{m} \frac{1}{|\beta_i|} \left( \frac{z}{f_i(z)} \right)^j \right| M^{|\beta|} + 1). \quad (2.3)\]
From (2.3) and (1.2), we see that
\[zC[f_1, f_2, \ldots, f_m]_{\beta_1, \beta_2, \ldots, \beta_m}(z)\]

From (2.4), and applying Lemma 1.2, we get
\[\left\|z \cdot \sum_{i=1}^{m} \frac{1}{|\beta_i|} \left( \frac{z}{f_i(z)} \right)^j \right\| M^{|\beta|} + 1 \leq 1.\]
Then \(T_{zC[f_1, f_2, \ldots, f_m]_{\beta_1, \beta_2, \ldots, \beta_m}(z)}\) is univalent in \(U\).

Corollary 2.1
Let \(f_i \in S\), for all \(i = 1, 2, \ldots, m\).

(1) If \(f_i\) are starlike of order \(\alpha\), then
\[\left\|z \cdot \sum_{i=1}^{m} \frac{1}{|\beta_i|} \left( \frac{z}{f_i(z)} \right)^j \right\| \leq 4 \sum_{i=1}^{m} \frac{1}{|\beta_i|} (1 - \alpha).\]
(2) If \(f_i\) are convex of order \(\alpha\), then
\[\left\|z \cdot \sum_{i=1}^{m} \frac{1}{|\beta_i|} \left( \frac{z}{f_i(z)} \right)^j \right\| \leq 2 \sum_{i=1}^{m} \frac{1}{|\beta_i|} (3 - 2\alpha).\]

Corollary 2. Let \(f_i \in S\).

(1) If \(f_i\) are starlike of order \(\alpha\), then
\[\left\|z \cdot \sum_{i=1}^{m} \frac{1}{|\beta_i|} \left( \frac{z}{f_i(z)} \right)^j \right\| \leq 4(1 - \alpha) \sum_{i=1}^{m} \frac{1}{|\beta_i|}.\]
(2) If \(f_i\) are convex of order \(\alpha\), then
\[\left\|z \cdot \sum_{i=1}^{m} \frac{1}{|\beta_i|} \left( \frac{z}{f_i(z)} \right)^j \right\| \leq 2(3 - 2\alpha) \sum_{i=1}^{m} \frac{1}{|\beta_i|}.\]

III. CONCLUSIONS
We conclude this study with some suggestions for future research; one direction is to obtain the order of convexity of the integral operator. Another direction would be studying other properties of the integral operator.

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