

## Comparison of Robust and Bayesian Methods for Estimating the Burr Type XII Distribution

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**Abstract**— This paper compares the robust and E-Bayesian estimations of the shape parameter for Burr XII distribution. BurrXII distribution was already reviewed by many researchers, as this distribution has gained special attention in recent times due to its complete applications, including the reliability field and failure time modeling. Burr distributions include 12 types of functions that produce a variety of probability density forms. We used two loss functions, quadratic, and LINEX with the E-Bayes method. The comparison conducted by simulation technique, and the absolute mean square error was measured to test the estimation methods' preference. In this study, many familiar distributions methods such as Weibull distribution, exponential logistic distribution, generalized logistic distribution, extreme value, and uniform distribution have been discussed accordingly by employing special cases and belonging to Burr distribution family. The current is dealing with the Bayesian method by depending on the parameter  $c$  that must be chosen to be close or not far from parameter  $b$  to ensure the robustness of the Bayesian estimator. Then the Bayesian expected of the parameter  $\beta$  under a quadratic loss function. It has been compared estimation for Burr-XII distribution by using the mentioned methods. We found many essential points, such as Robust estimates, in all cases, tend to be more efficient than the Bayes estimates. Also, by increasing the sample size, the robust estimates are still better than other estimation methods. When increasing the sample size, we notice a decrease in MAPE, which supports the statistical theory. We recommend using non-parametric methods to estimate Burr XII parameters. Thus, the main conclusion is that the robust process was the best.

**Keywords**— BurrXII distribution; bayesian estimation; e-bayesian estimation; robust; MAPE.

### I. INTRODUCTION

BurrXII distribution was first reviewed by Burr in 1942, as this distribution has gained special attention in recent times due to its wide applications, including reliability field and failure time modeling [1]. Burr distributions include 12 types of functions that produce a variety of probability density forms. Many familiar distributions such as Weibull distribution, exponential logistic distribution, generalized logistic distribution, extreme value, and uniform distribution are special cases and belong to Burr distribution family [2], [3].

The BurrXII distribution function is given by:

$$f(X; \gamma, \delta) = \gamma \delta X^{\gamma-1} (1 + X^\gamma)^{-(\delta+1)} \quad (1)$$

And the cumulative function

$$F(X; \gamma, \delta) = 1 - (1 + X^\gamma)^{-\delta} \quad (2)$$

Since  $\gamma, \delta$  are shape parameters.

From the distribution function (1), we note that this function equal:

$$f(0) = \begin{cases} \infty & \text{if } \gamma < 1 \\ \delta & \text{if } \gamma = 1 \\ 0 & \text{if } \gamma > 1 \end{cases}, \quad f(\infty) = 0$$

### II. MATERIALS AND METHOD

#### A. Bayesian Method

In this part, the parameter  $\beta$  will be estimated by the Bayesian method, assuming that the parameter  $\alpha$  is known [4][5]. The likelihood function is:

$$L(\gamma, \delta | \underline{X}) = \gamma^n \delta^n \prod_{i=1}^n X_i^{\gamma-1} \prod_{i=1}^n (1 + X_i^\gamma)^{-(\delta+1)} \quad (3)$$

Let the prior distribution of parameter  $\delta$  is the gamma distribution with the two parameters  $a_1, a_2$ :

$$\pi(\delta | a_1, a_2) = \frac{a_2^{a_1}}{\Gamma(a_1)} \beta^{a_1-1} \quad (4)$$

The posterior distribution of parameter  $\delta$  can be found from the product of the probability function of observations with the equations (3) and (4) according to the Bayes formula:

$$\pi(\delta|X) = \frac{\pi(\delta|a_1, a_2) * L(y, \delta|X)}{\int_0^\infty \pi(\delta|a_1, a_2) * L(y, \delta|X) d\delta} \quad (5)$$

### B. Bayes Estimator Using the Squared Loss Function

The Bayes estimator using the square loss function of the unknown parameter  $\delta$  is the mean of the posterior distribution function and gives as following [6], [7]:

$$\hat{\beta}_{BS} = E[\delta|X].$$

$$\hat{\beta}_{BS} = \int_0^\infty \delta \pi(\delta|X) d\delta \quad (6)$$

### C. Bayes Estimator Using LINEX Loss Function

The Bayes parameter of the parameter  $\delta$  using the LINEX type loss function is given as follows:

$$\hat{\beta}_{BL} = E[L(\delta)|X]$$

$$\hat{\beta}_{BL} = \int_0^\infty L(\delta) \pi(\delta|X) d\delta \quad (7)$$

### D. Empirical Bayesian

This section explains the E-Bayesian estimation parameter for the shape parameter  $\beta$  of the Burr XII distribution based on a square error and LINEX loss functions [10], [11]. The hyperparameters  $a, b$  must be chosen to ensure that the function  $\pi(\beta|a, b)$  in equation (7) must be decreasing in  $\beta$  where the derivative of  $\beta$  in this function is as follows:

Note that  $a > 0, b > 0, \beta > 0$  and  $\frac{d\pi(\beta|a, b)}{d\beta} < 0$  decreasing if  $0 < a < 1, b > 0$  and thus  $\pi(\beta|a, b)$  is a decreasing function, we will assume that  $a, b$  is independent with a binary distribution function.

$$\pi_j(a, b) = \pi(a)\pi(b) \quad j = 1, 2, 3$$

The empirical Bayes for the parameter  $\beta$  will be based on three different distributions of the hyperparameters  $a, b$ , and it used to study the effect of the different distributions above on the estimation of the predictive predictor of the parameter  $\beta$  [12].

Where the empirical Bayes for the parameter  $\beta$  will be as follows:

$$\hat{\beta}_{EB} = E[\beta_B|X] = \iint \hat{\beta}_B(a, b) \pi(a, b) d_a d_b \quad (8)$$

Where:  $\hat{\beta}_B(a, b)$  the Bayes estimator of the parameter  $\beta$  under the squared and LINEX loss functions. And the distributions of hyperparameters, uniform distribution, represent as follows:

$$\pi_1(a, b) = \frac{2a}{c} \quad 0 < a < 1, \quad 0 < b < c \quad (9)$$

$$\pi_2(a, b) = \frac{2b}{c^2} \quad 0 < a < 1, \quad 0 < b < c \quad (10)$$

$$\pi_3(a, b) = \frac{3b^2}{c^3} \quad 0 < a < 1, \quad 0 < b < c \quad (11)$$

The parameter  $c$  must be chosen to be close or not far from parameter  $b$  in order to ensure the robustness of the Bayesian estimator. Then the Bayesian expected of the parameter  $\beta$  under a quadratic loss function.

$$\hat{\beta}_{EBS_j} = \iint \hat{\beta}_{BS}(a, b) \pi_j(a, b) d_a d_b, \quad j = 1, 2, 3 \quad (12)$$

Also, the Bayesian expected of the parameter  $\beta$  can be found based on the LINEX type loss function.

$$\hat{\beta}_{EBL_j} = \iint \hat{\beta}_{BL}(a, b) \pi_j(a, b) d_a d_b \quad j = 1, 2, 3 \quad (13)$$

### E. Robust Method

We reviewed the least-squares method (LS), which is based on the estimation of the shape parameters  $\alpha, \beta$ , by minimizing the equation [8], [9]:

$$S = \sum_{i=1}^n (y_i - \log\beta - \log\log(1 + x_i^\alpha))^2 \quad (14)$$

Where:

$$F(x_{(i)}) = \frac{i-0.5}{n}, \quad i = 1, 2, \dots, n \quad (15)$$

When there are outlier values in the data, we will adjust the method of least squares (LS), the equation (14) will become as follows:

$$Q(\alpha, \beta) = \sum_{i=1}^n \rho(y_i - \log\beta - \log\log(1 + x_i^\alpha)) \quad (16)$$

So  $\rho$  represents the Huber function, which is written as follows:

$$\rho(x) = \begin{cases} x^2 & , |x| \leq b \\ 2b|x| - b^2 & , |x| > b \end{cases} \quad (17)$$

The derivative of the function (17) is as follows:

$$\rho'(x) = \begin{cases} x & , |x| \leq b \\ \text{sign}(x)b & , |x| > b \end{cases} \quad (18)$$

When the equation (16) is derived for the parameter  $\beta$

$$\log\hat{\beta} = \frac{\sum_{i=1}^n w_i y_i}{\sum_{i=1}^n w_i} - \frac{\sum_{i=1}^n w_i \log\log(1 + x_i^\alpha)}{\sum_{i=1}^n w_i} \quad (19)$$

Since  $w_i, i = 1, 2, \dots, n$  represents weights and it was written as follows:

$$w_i = \min \left\{ 1, \frac{b}{|y_i - \log\beta - \log\log(1 + x_i^\alpha)|} \right\} \quad (20)$$

If  $|y_i - \log\beta - \log\log(1 + x_i^\alpha)| \leq b$ , then the weights are equal to one, otherwise the weight will be

$$\frac{b}{|y_i - \log\beta - \log\log(1 + x_i^\alpha)|}$$

## III. RESULTS AND DISCUSSION

The simulation technique was conducted by using Matlab program; we applied this by taking prior information and the Steps of a simulation experiment as below:

First: Determinant prior values, size of the sample, and outlier ratio.

- Sample Size  $n$  (20, 30, 40).
- Hyper parameters ( $a = 0.8, b = 0.7, c = 1$ ).
- Known shape parameter  $\alpha$  (0.5, 1, 1.5).
- Outlier ratio (5%, 10%).
- Real value  $\beta$  (1)

Second: Generate data from Burr XII distribution.

Third: Comparison. In this step, the estimation methods of parameter  $\beta$  are compared by the Mean Absolute Parentage Error which is:  $\text{MAPE}(\hat{\beta}) = \frac{1}{M} \sum \left| \frac{\beta - \hat{\beta}}{\beta} \right|$ , where  $M = 1000$

represents the number of experiment repeats.

TABLE I  
REPRESENTING THE MAPE VALUES WITH 5% OUTLIERS

		BS	BL	EBS_F1	EBS_F2	EBL_F1	EBL_F2	Huber
20	$\alpha=0.5$ (a=0.8,b=0.7,c=1)	1.346458	1.043828	0.067044	0.006101	0.917967	0.065485	0.002478
	$\alpha=1$ (a=0.8,b=0.7,c=1)	1.292628	1.027841	0.777518	0.606682	0.945873	0.707867	6.06E-15
	$\alpha=1.5$ (a=0.8,b=0.7,c=1)	1.286429	1.029573	0.236786	0.06011	0.944146	0.223603	6.52E-14
30	$\alpha=0.5$ (a=0.8,b=0.7,c=1)	1.287047	1.028033	0.14006	0.022079	0.945602	0.135139	2.89E-18
	$\alpha=1$ (a=0.8,b=0.7,c=1)	1.286568	1.027456	0.082033	0.008066	0.946123	0.080643	5.56E-24
	$\alpha=1.5$ (a=0.8,b=0.7,c=1)	1.27886	1.02944	0.071766	0.006417	0.94399	0.070526	4.6E-24
40	$\alpha=0.5$ (a=0.8,b=0.7,c=1)	1.259784	1.017372	0.772057	0.597254	0.966562	0.729053	2.3E-05
	$\alpha=1$ (a=0.8,b=0.7,c=1)	1.253023	1.016756	0.238613	0.059432	0.96711	0.231136	1.31E-23
	$\alpha=1.5$ (a=0.8,b=0.7,c=1)	1.248271	1.016898	0.142553	0.021883	0.966924	0.13946	9.09E-30

TABLE II  
REPRESENTING THE MAPE VALUES WITH 10% OUTLIERS

	10% outliers	BS	BL	EBS_F1	EBS_F2	EBL_F1	EBL_F2	Huber
20	$\alpha=0.5$ (a=0.8,b=0.7,c=1)	1.522382	1.092913	0.821908	0.690598	0.838358	0.591516	0.536848
	$\alpha=1$ (a=0.8,b=0.7,c=1)	1.480394	1.086528	0.23008	0.064698	0.841209	0.195746	3.34E-05
	$\alpha=1.5$ (a=0.8,b=0.7,c=1)	1.484023	1.090054	0.134372	0.024461	0.839305	0.119507	1.22E-06
30	$\alpha=0.5$ (a=0.8,b=0.7,c=1)	1.52099	1.089437	0.075013	0.009189	0.841899	0.069888	0.03577
	$\alpha=1$ (a=0.8,b=0.7,c=1)	1.466726	1.097113	0.06785	0.007322	0.83453	0.063135	1.45E-08
	$\alpha=1.5$ (a=0.8,b=0.7,c=1)	1.323456	1.043367	0.788474	0.625763	0.917937	0.682337	6.74E-08
40	$\alpha=0.5$ (a=0.8,b=0.7,c=1)	1.336007	1.044447	0.233022	0.060027	0.917244	0.213898	1.34E-09
	$\alpha=1$ (a=0.8,b=0.7,c=1)	1.342053	1.040711	0.136204	0.022154	0.92061	0.129394	2.57E-12
	$\alpha=1.5$ (a=0.8,b=0.7,c=1)	1.340931	1.044699	0.079924	0.008288	0.917312	0.077327	4.77E-15

Note: The symbols in the tables above represent:

- BS : Bayes with a squared loss function
- BL : Bayes with LINEX loss function
- EBS\_F1 : Empirical Bayes with squared loss function and equation (9)
- EBS\_F2 : Empirical Bayes with squared loss function and equation (10)
- EBS\_F3 : Empirical Bayes with squared loss function and equation (11)
- EBL\_F1 : Empirical Bayes with LINEX loss function and equation (9)
- EBL\_F2 : Empirical Bayes with LINEX loss function and equation (10)
- EBL\_F3 : Empirical Bayes with LINEX loss function and equation (11)
- Huber : Robust with Huber

#### IV. CONCLUSION

We compared the estimation for Burr-XII distribution by using the mentioned methods, and we found the following results: Robust estimates, in all cases, tend to be more efficient than the Bayes estimates. By increasing the size of sample, the robust estimates still better than other estimation method. When increasing the sample size, we notice a decrease in MAPE, which supports the statistical theory. We recommend using non-parametric methods to estimate Burr XII parameters.

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